Project 1

You can use MATLAB, Python, C/C++, FORTRAN or any other programming language to write your code. Submit your project report as a single pdf file (named as "*lastname_ firstname_ Project1.pdf*") on Carmen. Also, submit a copy of your code (in a zipped file, named as "*lastname_ firstname_ Project1.zip*") through Carmen.

1. (10 pts). Projections
Write a function $proj(A, b)$ which projects vector b in the column space of matrix A when:

(a) A has only linearly independent columns
(b) (optional) A might have linearly dependent columns

Find the projection of $b = (2, 1, 3, 1, 4)$ onto the column space of

$$A = \begin{bmatrix} 1 & 1 & 0 & (1-\varepsilon) \\ 1 & 0 & 1 & \varepsilon + (1-\varepsilon) \\ 0 & 1 & 0 & \varepsilon \\ 1 & 0 & 0 & (1-\varepsilon) \\ 1 & 0 & 0 & \varepsilon + (1-\varepsilon) \end{bmatrix}.$$

for $\varepsilon = 1$. What happens when $\varepsilon \to 0$? Study this case numerically.

2. (50 points pts). QR factorisation.

(a) Implement the classical Gram-Schmidt (CGS) algorithm, modified Gram-Schmidt (MGS) algorithm, and Householder QR factorization by writing your own codes. The code should work for any matrix sized $m \times n$, when $m \geq n$. Each method should have a corresponding function that takes as input the matrix A and outputs the orthogonal matrix Q. (Note that the Householder algorithm does not output Q automatically, and you need to compute Q explicitly.)
(b) The Hilbert matrix has entries $h_{ij} = 1/(i+j-1)$. For example, the 2×2 Hilbert matrix has entries:

$$\begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix}.$$

Use each algorithm in (a) to generate an orthogonal matrix Q whose columns form an orthonormal basis for the column space of the Hilbert matrix $H \in \mathbb{R}^{n \times n}$, for $n = 2, ..., 20$.
(c) Consider a fourth method which takes the Q generated by the CGS method and applies the CGS method again and use that to find an orthonormal basis for the column space of the Hilbert matrix $H \in \mathbb{R}^{n \times n}$, for $n = 2, ..., 20$.
(d) Quantify the quality of results of the 4 methods above (in (a) and (c)) by plotting the quantity $\log_{10}(||I - Q^T Q||_F)$ as a function of n, which measures potential loss of orthogonality of the matrix Q due to computational error. In addition to the plot, report the results in a table when $n = 4, n = 9, n = 11$ in a table.
(e) How to the four methods compare in **speed** and **accuracy**? Write down what you observe and try to explain the observations.

3. (40 pts). Least square problem.
Consider the function

$$f(t) = \frac{1}{1+t^2} \quad t \in [-5, 5].$$

For a given positive integer N, divide the interval $[-5, 5]$ into N equally spaced subintervals. Let $\{t_j\}_{j=0}^{N}$ denote the endpoints of the subintervals in decreasing order, with $t_0 = -5$ and $t_N = 5$. Denote $y_t = f(t_j)$ as the function value at point x_j. Let $n \leq N$ be another positive integer and write

$$p(t) = a_n t^n + a_{n-1} t^{n-1} + ... + a_1 t + a_0$$

with $n+1$ coefficients $\{a_j\}$. For $\{(t_j, y_j)\}_{j=0^N}$ and $p(t)$ we introduce the multivariate function

$$J(a_0, a_1, ..., a_n) := \sum_{j=0}^{N} |y_j - p(t_j)|^2.$$

The least square data fitting problem of the dataset $\{(t_j, y_j)\}_{j=0^N}$ by the $n-$th order polynomial $p(t)$ is:

$$\arg \min_{a_0,...,a_n} J(a_0, a_1, ..., a_n).$$

(a) Reformulate this problem as a least square problem for $Ax = b$ by introducing the appropriate A, b and x.
(b) Use the Householder triangulation to solve this problem.
(c) Consider the pairs $N = 30, n = 5$, $N = 30, n = 15$, $N = 30, n = 30$. Plot both functions $f(t)$ and $p(t)$ in the same plot for each of the pairs above. What do you observe?
(d) Will p converge to f if $N = n$ tends to infinity? You can use numerical experiments to justify your answer.
(e) Now consider f to be given by

$$f(t) = t^n + t^{n-1} + ... + t + 1 \quad t \in [0, 1]$$

Repeat (a) and (b) choosing $n = 4, 6, 8, 10, ..., 20$ and $N = 2n$. Plot the quantity $\log_{10} ||(a_0 - 1, ..., a_n - 1)||_2$, which measures the error between the found coefficients and the true coefficients (which are all 1). What do you observe?